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A Modified Harmonic Balance Method for Solving Strongly Generalized Nonlinear Damped Forced Vibration Systems

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Abstract: A modified harmonic balance method is proposed for solving damped forced generalized nonlinear oscillators with strong nonlinearity. In the classical harmonic balance method, a set nonlinear algebraic equation of the unknown coefficients is solved by the numerical method to determine the unknown coefficients. However, in the present method, only one nonlinear equation and a set of linear algebraic equations are required for solution, thereby reducing the computational effort. Comparison between the results obtain by the proposed method and the numerical method is presented in figures which show a good agreement with the numerical results. The proposed method can play an important role for handling such nonlinear dynamical systems.

Keywords: *harmonic balance method; generalized nonlinear damped oscillators; forcing term.*

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1 Introduction

Nonlinear oscillators are very important in many areas of applied mathematics, physics, and engineering. Most of the physical problems are governed by the nonlinear differential equations. The exact solutions of these nonlinear equations are rarely obtained. Therefore, many researchers focused on analytical approximation methods. Among them, the perturbation method [1,2], homotopy analysis method [3,4], homotopy perturbation method [5–7], variational iteration technique [8,9], harmonic balance method (HBM)[10–13] are well known. The perturbation methods [14–20] are widely used techniques for dealing with the nonlinear differential systems and they were originally developed for weakly nonlinear dynamical systems. Jones [20] modified the perturbation method to extend the accuracy of the solution when the parameter was not small. Further, a modification of the Lindstedt-Poincare technique was presented by Cheung et al. [21] based on the Jones technique [20. The modified Lindstedt-Poincare method has been generalized by Alam, et al.[22] and it is applicable for a wide variety of nonlinear oscillators. The harmonic balance method (HBM) is another powerful technique to obtain the periodic solution of the nonlinear oscillators. According to this method, the solution is chosen as a truncated Fourier series. Usually, a set of strongly nonlinear algebraic equations appears among the unknown coefficients of several harmonic terms and these equations are solved by the numerical method. Further, this method has been modified by several researchers [23–28]. Rahman et al. [23] used a modified HBM to study the Van der Pol equation. Rahman and Lee [26] developed a modified residue HBM to handle nonlinear vibrating problems of beam. Wu [27] developed the harmonic balance method for the Yao-Cheng oscillator. Wagner and Lentz [28] developed a HBM to handle the Duffing oscillator with a forcing term with cubic nonlinearity. Younesian et al. [29] applied He's frequency-amplitude formulation and He's energy balance method to handle strongly nonlinear the generalized Duffing oscillators without forcing term. Uddin et al.[30] presented an analytical approximation technique for handling the generalized nonlinear Duffing equation with strong nonlinearity without external forcing term. Rafieipour et al. [31] developed an analytical approximate solution for the generalized nonlinear vibration of a micro electro mechanical system by using He's frequency amplitude formulation. Karahan and Pakdemirli [32] studied free and forced vibration response of the strongly nonlinear cubic-quintic Duffing oscillators by using the multiple time scale method. Ullah et al. [33] developed a modified harmonic balance method to handle nonlinear oscillators with cubic nonlinearity in the presence of external forcing term. Rahman et al. [34] presented a modified harmonic balance method to solve the nonlinear vibration problem of a beam resting on nonlinear foundation. Recently, Yeasmin et al. [35] have presented an analytical technique for handing the quadratic nonlinear oscillator based on the harmonic balance method for free vibration nonlinear problems. Cheib et al. [36] presented an analysis of the dynamics of a two-degree-of-freedom nonlinear mechanical system under harmonic excitation. It is noticed that the approximate analytical techniques for solving the damped forced generalized nonlinear oscillators with strong nonlinearity are almost untouched. To fill this gap, a modified harmonic balance method has been presented for handling strongly generalized nonlinear damped forced oscillators. The convenience of this method is that only one nonlinear algebraic equation and a set of linear algebraic equations are required to solve by the numerical method, which reduces the heavy computational effort that is required in classical harmonic balance methods. The obtained results are compared with the corresponding numerical results in graphs and it shows a good agreement with these numerical results.

2 Method

Let us assume a strongly generalized nonlinear damped forced oscillator [29–33] of the form

$$\ddot{x} + \mu\dot{x} + \omega_0^2 x + \epsilon(\alpha_3 f_3(x) + \alpha_5 f_5(x) + ... + \alpha_n f_n(x)) = E\ \cos(\omega t), \tag{1}$$

where over-dots denote differentiation with respect to time t, ω_0 is the natural frequency, μ is the linear damping coefficient, $f_i(x)$ are given general nonlinear functions of x, α_i $(i = 1, 3, 5, ...n)$ are constants, ϵ is a positive parameter which is not necessarily small, E is the amplitude of the excitation force and ω is the forcing frequency. All of the parameters are positive. We assume that $\mu = 0$ in our idealized systems. But damped motion is important for most of the physical and engineering vibration problems. In this paper, we are going to assume that $\mu \neq 0$. This is a non-autonomous system since time t explicitly appears in the right-hand side of the given equation. In particular, periodically forced harmonic oscillators depended explicitly on time t and exhibited quite interesting behavior. When a damped Duffing-type oscillator is driven with a periodic forcing function, the result may be a periodic response at the same frequency as the forcing function. Since the unforced oscillation is the dissipated energy due to the damping, we are not surprised to find that it is absent from the steady state forced behavior. According to the proposed method, the approximate solution of Eq.(1) is assumed [33] in the following form:

$$x = a\ \cos(\omega t) + b\ \sin(\omega t) + a_3\ \cos(3\omega t) + b_3\ \sin(3\omega t) + ..., \tag{2}$$

where a, b, a_3,b_3... are the unknown coefficients. Now, differentiating Eq.(2) twice with respect to t and then putting into Eq.(1) and expanding $f_i(x)$ as a truncated Fourier series expansion and taking the coefficients of equal harmonics from both sides, we obtain the following set of algebraic equations:

$$a(-\omega^2 + \omega_0^2) + b\mu\omega + \epsilon C_1(a, b, a_3, b_3, ...) = E, \tag{3a}$$

$$a(-\omega^2 + \omega_0^2) - a\mu\omega + \epsilon S_1(a, b, a_3, b_3, ...) = 0, \tag{3b}$$

$$a_3(-9\omega^2 + \omega_0^2) + 3b_3\mu\omega + \epsilon C_3(a, b, a_3, b_3, ...) = 0, \tag{3c}$$

$$b_3(-9\omega^2 + \omega_0^2) - 3a_3\mu\omega + \epsilon S_3(a, b, a_3, b_3, ...) = 0. \tag{3d}$$

Eliminating ω^2 from the Eqs.(3b)-(3d) with the help of Eq.(3a), we get

$$\omega^2 = \omega_0^2 + b\mu\omega/a + \epsilon C_1(a, b, a_3, b_3, ...) - E/a, \tag{4a}$$

$$-b^2\mu\omega/a - a\mu\omega - \epsilon b C_1(a, b, a_3, b_3, ...) + \epsilon S_1(a, b, a_3, b_3, ...) + bE/a = 0, \tag{4b}$$

$$-8\omega_0^2 a_3 + 3b_3\mu\omega - 9a_3 b\mu\omega/a - 9\epsilon a_3 C_1(a, b, a_3, b_3, ...) + \epsilon C_3(a, b, a_3, b_3, ...) + 9a_3 E/a = 0, \tag{4c}$$

$$-8\omega_0^2 b_3 - 3a_3\mu\omega - 9b_3 b\mu\omega/a - 9\epsilon b_3 C_1(a, b, a_3, b_3, ...) + \epsilon S_3(a, b, a_3, b_3, ...) + 9b_3 E/a = 0. \tag{4d}$$

Now, using Eq.(4b), eliminating ω from the Eqs.(4c)-(4d) and taking only the linear terms of a_3, b_3 and neglecting the terms of insignificant effects, we obtain two linear equations for a_3 and b_3. From these equations a_3 and b_3 are determined. After putting

a_3 and b_3 into Eq.(4b), b is expressed as a power series of small parameter $\lambda(\mu, \omega, E)$ in the following form:

$$b = m_0 + m_1\lambda + m_2\lambda^2 + m_3\lambda^3 + ..., \tag{5}$$

where m_0, m_1, m_2 are the functions of a. Finally, after putting a_3, b_3 and b into Eq.(4a) and then solving this equation, the values of a are determined. Consequently, the desired values of b, a_3 and b_3 are calculated.

3 Example

Consider a generalized nonlinear (cubic-quintic) damped forced oscillator [29-33] of the following form:

$$\ddot{x} + \mu\dot{x} + x + \epsilon(\alpha_3 x^3 + \alpha_5 x^5) = E\,\cos(\omega t), \tag{6}$$

where ω_0^2=1. According to the truncated Fourier series, the solution of Eq.(6) is assumed as [33]

$$x = a\,\cos(\omega t) + b\,\sin(\omega t) + a_3\,\cos(3\omega t) + b_3\,\sin(3\omega t) + ... \tag{7}$$

Putting Eq.(7) with its derivatives into Eq.(6) and then equating the coefficients of equal harmonics on both sides, we obtain

$$\begin{aligned}&16(a + b\mu\omega - a\omega^2) + 12\epsilon((a^2 - b^2)a_3 + 2aa_3^2 + a(a^2 + b^2 + 2bb_3 + 2b_3^2)\\&+ 5\epsilon(6(a^2 - b^2)a_3^3 + 6aa_3^4 + 12aa_3^2(a^2 + b^2 + bb_3 + b_3^2) + a_3(5a^4 - 6a^2b^2 - 3b^4\\&+ 6(a^2 - b^2)b_3^2) + 2a((a^2 + b^2)^2 + 2(3a^2b + b^3)b_3 + 6(a^2 + b^2)b_3^2 + 6bb_3^3\\&+ 3b_3^4))\alpha_5 = 16E,\end{aligned} \tag{8a}$$

$$\begin{aligned}&- 16(a\mu\omega + b(-1 + \omega^2)) + 12\epsilon(b(a^2 + b^2) - 2aba_3 + 2ba_3^2 + (a^2 - b^2)b_3 + 2bb_3^2)\alpha_3\\&+ 5\epsilon(2b(a^2 + b^2)^2 - 12aba_3^3 + 6ba_3^4 + (3a^4 + 6a^2b^2 - 5b^4)b_3 + 12b(a^2 + b^2)b_3^2\\&+ 6(a^2 - b^2)b_3^3 + 6bb_3^4 - 4aba_3(a^2 + 3b^2 + 3b_3^2) + 6a_3^2(2b(a^2 + b^2)\\&+ (a^2 - b^2)b_3 + 2bb_3^2))\alpha_5 = 0,\end{aligned} \tag{8b}$$

$$\begin{aligned}&48\mu\omega b_3 + \epsilon(30a(a^2 - 3b^2)a_3^2\alpha_5 + 10a_3^5\alpha_5 + 10a(a^2 - 3b^2)b_3^2\alpha_5 + a(a^2 - 3b^2)(4\alpha_3 +\\&5(a^2 + b^2)\alpha_5) + 4a_3^3((3\alpha_3 + 5(3(a^2 + b^2) + b_3^2)\alpha_5)) + 2a_3(8 - 72\omega^2 + 6\epsilon(2(a^2 + b^2)\\&+ b_3^2)\alpha_3) + 5\epsilon(3(a^2 + b^2)^2 + (6a^2b - 2b^3)b_3 + 6(a^2 + b^2)b_3^4)\alpha_5) = 0,\end{aligned} \tag{8c}$$

$$\begin{aligned}&10\epsilon a_3^4b_3\alpha_5 + \epsilon(-30b(-3a^2 + b^2)b_3^2\alpha_5 + 10b_3^5\alpha_5 + 12b_3^3(\alpha_3 + 5(a^2 + b^2)\alpha_5))\\&- b(-3a^2 + b^2)(4\alpha_3 + 5(a^2 + b^2)\alpha_5)) + 2b_3(8 - 72\omega^2 + 12\epsilon(a^2 + b^2)\alpha_3\\&+ 15\epsilon(a^2 + b^2)^2\alpha_5) - 4a_3(12\mu\omega - 5\epsilon a(a^2 - 3b^2)b_3\alpha_5) + 2\epsilon a_3^2(-5b(-3a^2 + b^2)\alpha_5\\&+ 10b_3^3\alpha_5 + 6b_3(\alpha_3 + 5(a^2 + b^2)\alpha_5)) = 0.\end{aligned} \tag{8d}$$

Eliminating ω^2 from the Eqs.(8b)-(8d) with the help of Eq.(8a), and ignoring the terms whose responses are negligible, we obtain the following equations:

$$\begin{aligned}&- 16(-bE + a^2\mu\omega + b^2\mu\omega) - 3\epsilon(ba_3(3a^2 - b^2)(4\alpha_3 + 5(a^2 + b^2)\alpha_5)\\&- ab_3(a^2 - 3b^2)(4\alpha_3 + 5(a^2 + b^2)\alpha_5)) = 0,\end{aligned} \tag{9a}$$

$$\begin{aligned}&-4a_3(4(8a-9E+9b\mu\omega)+\epsilon a(21(a^2+b^2)\alpha_3+15(a^2+b^2)^2\alpha_5))\\&+a(48\mu\omega b_3+\epsilon a(a^2-3b^2)(4\alpha_3+5(a^2+b^2)\alpha_5))=0,\end{aligned} \tag{9b}$$

$$\begin{aligned}&-48a\mu\omega a_3+\epsilon ab(3a^2+b^2)(4\alpha_3+5(a^2+b^2)\alpha_5)-4b_3(4(8a-9E+9b\mu\omega)\\&+\epsilon a((21(a^2+b^2)\alpha_3+15(a^2+b^2)\alpha_5))=0.\end{aligned} \tag{9c}$$

Now, using Eq.(9b), eliminating ω from the Eqs.(9c) and (9d) and taking only the linear terms of a_3, b_3 and omitting the terms whose response is negligible, we obtain

$$\begin{aligned}&\epsilon a(a^4-2a^2b^2-3b^4)(4\alpha_3+5(a^2+b^2)\alpha_5)-4a_3(4(8a^2+8b^2-9aE)\\&+\epsilon(21(a^2+b^2)^2\alpha_3+15(a^2+b^2)^3\alpha_5))=0,\end{aligned} \tag{10a}$$

$$\begin{aligned}&-\epsilon b(3a^4+2a^2b^2-b^4)(4\alpha_3+5(a^2+b^2)\alpha_5)+4b_3(4(8a^2+8b^2-9aE)\\&+\epsilon(21(a^2+b^2)^2\alpha_3+15(a^2+b^2)^3\alpha_5))=0.\end{aligned} \tag{10b}$$

After solving Eqs.(10a) and (10b), a_3 and b_3 are determined as follows:

$$\begin{aligned}&a_3=\epsilon a(a^4-2a^2b^2-3b^4)(4\alpha_3+5(a^2+b^2)\alpha_5)\\&/4(4(8a^2+8b^2-9bE)+\epsilon(21(a^2+b^2)^2)\alpha_3+15(a^2+b^2)^3\alpha_5),\end{aligned} \tag{11a}$$

$$\begin{aligned}&b_3=\epsilon b(3a^4+2a^2b^2-b^4)(4\alpha_3+5(a^2+b^2)\alpha_5)\\&/4(4(8a^2+8b^2-9bE)+\epsilon(21(a^2+b^2)^2)\alpha_3+15(a^2+b^2)^3\alpha_5).\end{aligned} \tag{11b}$$

Inserting the values of a_3 and b_3 into Eq.(9a), we then expand b in a power series of the small parameter λ as follows:

$$b=l_0+l_1\lambda+l_2\lambda^2+l_3\lambda^3+...,\tag{12}$$

where $\lambda=2\mu\omega/E$, $l_0=a^2\mu\omega/E$, $l_1=a^4\mu^2\omega^2/E^2$, $l_2=2a^6\mu^3\omega^3/E^3$, ... Finally,upon inserting a_3, b_3 and b into Eq.(8a) and solving, the values of a are obtained. Consequently, the values b, a_3 and b_3 are determined.

4 Results and Discussion

The solutions determined by the present technique are compared with the corresponding numerical solution to justify the validity and the accuracy of the proposed technique. Comparisons between the solution curves obtained by the proposed method and a numerical method are shown graphically in Figures 1-4 in the presence of various damping and different values of the system parameters for strongly generalized nonlinear forced vibration problems.

From the figures, it is seen that the approximate results agree nicely with those solutions obtained by the numerical procedure.

5 Conclusion

In this study, a modified harmonic balance method is presented for handling strongly generalized nonlinear damped forced vibration problems. Some limitations of the classical HBM are overcome by the proposed method. The advantage of the present technique is that only one nonlinear algebraic equation is needed for solution. As a result, the computational effort is reduced and less effort is required than in the existing classical

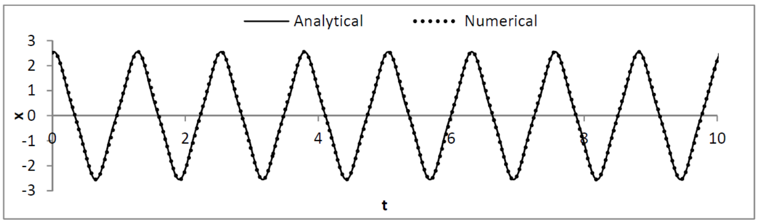

Figure 1: Comparison between the results obtained by the presented method and a numerical technique when $\omega = 5, \epsilon = 1.0, \alpha_3 = 1, \alpha_5 = 1, \mu = 0.1, E = 10$.

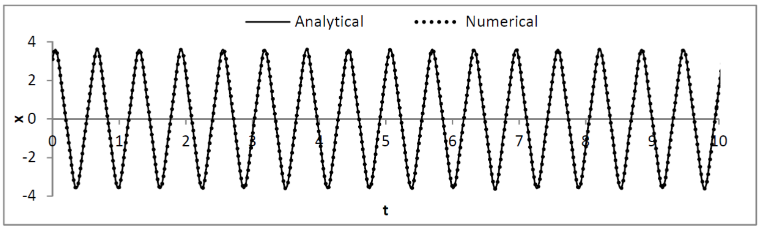

Figure 2: Comparison between the results obtained by the presented method and a numerical technique when $\omega = 10, \epsilon = 1.0, \alpha_3 = 1, \alpha_5 = 1, \mu = 0.25, E = 20$.

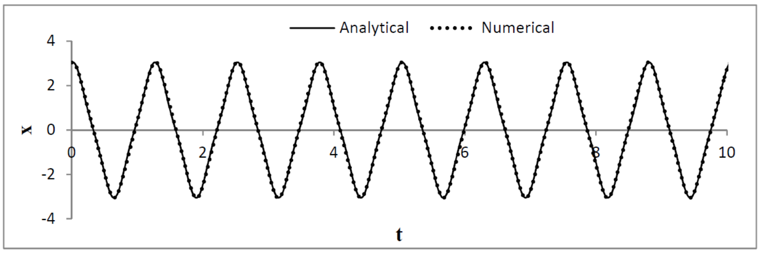

Figure 3: Comparison between the results obtained by the presented method and a numerical technique when $\omega = 5, \epsilon = 0.5, \alpha_3 = 1, \alpha_5 = 1, \mu = 0.05, E = 10$.

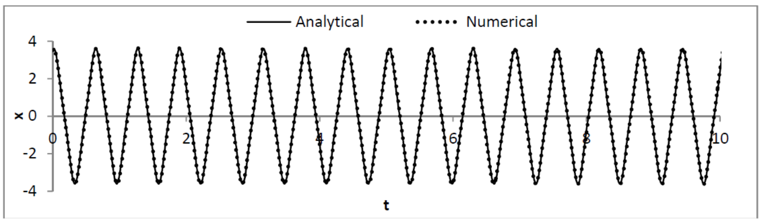

Figure 4: Comparison between the results obtained by the presented method and a numerical technique when $\omega = 10, \epsilon = 1.0, \alpha_3 = 1, \alpha_5 = 1, \mu = 0.1, E = 20$.

harmonic balance method. The results obtained by the present method show a good agreement with the numerical results. It is assumed that the proposed method is very effective and convenient for damped forced generalized nonlinear oscillators with strong as well as weak nonlinearities. Our results exhibit acceptable complaince with the solutions computed by the fourth order Runge-Kutta method for several values of systems parameters and significant damping.

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